# Harmonijski oscilator

– Rešenje svojstvenog problema LHO-a dato je tzv. Ermitovim f-jama

$$\psi_n(\xi) = c_n \, e^{-\frac{\xi^2}{2}} H_n(\xi)$$

gde su $H_n(\xi)$ tzv. Ermitovi polinomi, $\xi = \sqrt{\alpha}\, x$, $\alpha = m\omega/\hbar$, $x \in (-\infty, +\infty)$. Ovaj skup f-ja predstavlja ortonormirani bazis, tj.

$$\int_{-\infty}^{+\infty} \psi_n^*(\xi)\, \psi_m(\xi)\, d\xi = \delta_{nm}$$

– Svojstvene vrednosti za energiju su date izrazom

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \ldots$$

– Rekurentne relacije za Ermitove f-je.

$$\xi \psi_n(\xi) = \sqrt{\frac{n}{2}}\, \psi_{n-1}(\xi) + \sqrt{\frac{n+1}{2}}\, \psi_{n+1}(\xi)$$

$$\frac{d}{d\xi}\psi_n(\xi) = \sqrt{\frac{n}{2}}\, \psi_{n-1}(\xi) - \sqrt{\frac{n+1}{2}}\, \psi_{n+1}(\xi)$$

– F-ja generatrisa Ermitovih polinoma

$$e^{-z^2 + 2z\xi} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(\xi)$$

$\Longrightarrow$

1. Eksplicitnim računom dokazati konačnost Ermitovih f-ja jedne promenljive u beskonačnosti.

$$\lim_{x\to\pm\infty} \psi_n(x) = 0$$

$$\boxed{\psi_n(\xi) = C_n H_n(\xi)\, e^{-\xi^2/2} \qquad \xi = \sqrt{\alpha}\, x \qquad \alpha = \frac{m\omega}{\hbar}}$$

$$\psi_n(\xi) = C_n \frac{H_n(\xi)}{e^{\xi^2/2}} \quad ; \quad H_n(\xi) = \sum_{i=0}^{n} c_i \xi^i$$

Ermitov polinom

$$\lim_{\xi\to\pm\infty} \psi_n(\xi) = \frac{\infty}{\infty}$$

Mora se primeniti n puta Lopitalovo pravilo da bi se polinom doveo na konstantu. Onda će biti

$$\lim_{\xi\to+\infty} \psi_n(\xi) \approx \lim_{\xi\to\infty} C_n \frac{c_0}{e^{\xi^2/2}} = 0$$

$$\lim_{\xi\to-\infty} \psi_n(\xi) = 0$$ ~~jer~~ $\cancel{\lim_{\xi\to\pm\infty} H_n(\xi) = 0}$

Ermitove f-je grade Hilbertov prostor

$$\mathcal{H} = \left\{ \psi_n(x) \;\middle|\; \int_{-\infty}^{+\infty} |\psi_n(x)|^2 dx < \infty \right\} \Leftrightarrow$$

$$\lim_{x\to\pm\infty} \psi_n(x) = 0$$

2. Naći očekivane vrednosti opservabli

a) $\hat{x}, \hat{x}^2$ ; b) $\hat{x}^3, \hat{x}^4, \hat{x}^5, \hat{x}^6$ – za ispit

$\boxed{\text{Domaći}}$

u svojstvenom stanju LHO-a, određenog kvantni brojem $n$.

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Koristi se rekurentna relacija

$$\xi \psi_n(\xi) = \sqrt{\frac{n+1}{2}}\, \psi_{n+1}(\xi) + \sqrt{\frac{n}{2}}\, \psi_{n-1}(\xi)$$

$$\xi = \sqrt{\alpha} \cdot x \qquad , \quad \alpha = m\omega/\hbar$$

a)

$$\langle \hat{x} \rangle = \int_{-\infty}^{+\infty} \psi_n^*(\xi)\, x\, \psi_n(\xi)\, d\xi \qquad [\psi_n^*(\xi) = \psi_n(\xi)]$$

$$= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{+\infty} \psi_n(\xi)\, \underbrace{\xi \psi_n(\xi)}\, d\xi$$

$$= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{+\infty} \psi_n(\xi) \left( \sqrt{\frac{n+1}{2}}\, \psi_{n+1}(\xi) + \sqrt{\frac{n}{2}}\, \psi_{n-1}(\xi) \right) d\xi$$

$$= \frac{1}{\sqrt{\alpha}} \left[ \int_{-\infty}^{+\infty} \sqrt{\frac{n+1}{2}}\, \psi_n(\xi)\, \psi_{n+1}(\xi)\, d\xi + \int_{-\infty}^{+\infty} \sqrt{\frac{n}{2}}\, \psi_n(\xi)\, \psi_{n-1}(\xi)\, d\xi \right]$$

$$= 0$$

$$\langle \hat{x}^2 \rangle = \int_{-\infty}^{+\infty} \psi_n^*(\xi)\, x^2\, \psi_n(\xi)\, d\xi =$$

$$= \frac{1}{\alpha} \int_{-\infty}^{+\infty} \psi_n(\xi)\, \xi^2\, \psi_n(\xi)\, d\xi =$$

$$= \frac{1}{\alpha} \int_{-\infty}^{+\infty} \psi_n(\xi)\, \xi \left( \sqrt{\frac{n+1}{2}}\, \psi_{n+1}(\xi) + \sqrt{\frac{n}{2}}\, \psi_{n-1}(\xi) \right) d\xi$$

$$= \frac{1}{\alpha} \left[ \sqrt{\frac{n+1}{2}} \int_{-\infty}^{+\infty} \underbrace{\psi_n(\xi)\, \xi}\, \psi_{n+1}(\xi)\, d\xi + \sqrt{\frac{n}{2}} \int_{-\infty}^{+\infty} \underbrace{\psi_n(\xi)\, \xi}\, \psi_{n-1}(\xi)\, d\xi \right]$$

$$= \frac{1}{\alpha} \left[ \sqrt{\frac{n+1}{2}} \int_{-\infty}^{+\infty} \left( \sqrt{\frac{n+1}{2}}\, \psi_{n+1}(\xi) + \sqrt{\frac{n}{2}}\, \psi_{n-1}(\xi) \right) \psi_{n+1}(\xi)\, d\xi + \sqrt{\frac{n}{2}} \int_{-\infty}^{+\infty} \left( \sqrt{\frac{n+1}{2}}\, \psi_{n+1}(\xi) + \sqrt{\frac{n}{2}}\, \psi_{n-1}(\xi) \right) \psi_{n-1}(\xi)\, d\xi \right]$$

$$= \frac{1}{\alpha} \left[ \frac{n+1}{2} + \frac{n}{2} \right] = \frac{2n+1}{2\alpha}$$

Za domaći $\langle \hat{x}^3 \rangle = ?$ $\quad (\langle \hat{x}^3 \rangle = 0)$

Rešiti sv. problem ~~H~~ hamiltonijana

$$\hat{H} = \frac{\hat{P}_x^2}{2m} + \frac{m\omega^2}{2}(\hat{x}-x_0)^2$$

gde $x_0$ predstavlja realan broj sa dimenzijom dužine. Dokazati da važi $\langle\psi_n|\hat{x}|\psi_n\rangle = x_0$, gde su $|\psi_n\rangle$-ovi svojst. f-je gornjeg hamiltonijana.

U koordinatnoj reprezentaciji

$$H = -\hbar^2 \frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}(x-x_0)^2$$

$$x - x_0 = z$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} = \frac{\partial}{\partial x}$$

$$H = -\hbar^2 \frac{\partial^2}{\partial z^2} + \frac{m\omega^2}{2} z^2$$

$$\xi = \sqrt{\alpha}\, z = \sqrt{\alpha}\,(x-x_0)$$

$$\downarrow$$

$$\psi_n(\xi) = C_n H_n(\xi)\, e^{-\frac{\xi^2}{2}}, \quad E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega$$

$$\langle\psi_n|\hat{z}|\psi_n\rangle = 0$$ (neni od prethodnih zadataka)

$$\Downarrow$$

$$\langle\psi_n|\hat{x}|\psi_n\rangle = x_0 \longrightarrow \text{provera}$$

$$\int_{-\infty}^{+\infty} \psi_n^*(x-x_0)\, x\, \psi_n(x-x_0)\, dx = \left[\text{Smena } \begin{matrix} x-x_0 = t \\ dx = dt \end{matrix}\right]$$

$$\int_{-\infty}^{+\infty} \psi_n(t)(t+x_0)\,\psi_n(t)\, dt$$

$$\overset{0}{\cancel{\int_{-\infty}^{+\infty} \psi_n(t)\, t\, \psi_n(t)\, dt}} + x_0 \int_{-\infty}^{+\infty} |\psi_n(t)|^2\, dt = x_0$$

4. Izračunati $\langle n|\hat{x}\hat{p}_x|n\rangle$, gde je $|n\rangle$ svojstveno stanje LHO-a. Na osnovi ovoga izračunati $\langle n|\hat{p}_x\hat{x}|n\rangle$.

$$\langle n|\hat{x}\hat{p}_x|n\rangle = -i\hbar\int_{-\infty}^{+\infty}\psi_n^*(\xi)\, x\,\frac{d}{dx}\psi_n(\xi)\, d\xi$$

$$\xi = \sqrt{\alpha}\, x \Rightarrow x = \frac{\xi}{\sqrt{\alpha}}$$

$$\frac{d}{dx} = \frac{d}{d\xi}\frac{d\xi}{dx} = \sqrt{\alpha}\frac{d}{d\xi}$$

$$\langle n|\hat{x}\hat{p}_x|n\rangle = -i\hbar\int_{-\infty}^{+\infty}\psi_n(\xi)\,\frac{\xi}{\sqrt{\alpha}}\sqrt{\alpha}\,\frac{d}{d\xi}\psi_n(\xi)\, d\xi$$

$$= -i\hbar\int_{-\infty}^{+\infty}\psi_n(\xi)\,\xi\,\frac{d\psi_n(\xi)}{d\xi}\, d\xi$$

$$= -i\hbar\int_{-\infty}^{+\infty}\psi_n(\xi)\,\xi\left(\sqrt{\frac{n}{2}}\psi_{n-1}(\xi) - \sqrt{\frac{n+1}{2}}\psi_{n+1}(\xi)\right)d\xi$$

$$\xi\psi_n(\xi) = \sqrt{\frac{(n+1)}{2}}\psi_{n+1}(\xi) + \sqrt{\frac{n}{2}}\psi_{n-1}(\xi)$$

$$\frac{d}{d\xi}\psi_n(\xi) = \sqrt{\frac{n}{2}}\psi_{n-1}(\xi) - \sqrt{\frac{(n+1)}{2}}\psi_{n+1}(\xi)$$

$$\cancel{= -i\hbar\int_{-\infty}^{+\infty}\psi_n(\xi)}$$

$$= -i\hbar\left[\sqrt{\frac{n}{2}}\int_{-\infty}^{+\infty}\psi_n(\xi)\,\xi\,\psi_{n-1}(\xi)\,d\xi - \sqrt{\frac{n+1}{2}}\int_{-\infty}^{+\infty}\psi_n(\xi)\,\xi\,\psi_{n+1}(\xi)\,d\xi\right]$$

$$= -i\hbar \left[ \sqrt{\frac{n}{2}} \int_{-\infty}^{+\infty} \psi_n(\xi) \left( \sqrt{\frac{n}{2}}\, \psi_n(\xi) + \sqrt{\frac{n-2}{2}}\, \psi_{n-2}(\xi) \right) d\xi \right.$$

$$\left. - \sqrt{\frac{n+1}{2}} \int_{-\infty}^{+\infty} \psi_n(\xi) \left( \sqrt{\frac{n+2}{2}}\, \psi_{n+2}(\xi) + \sqrt{\frac{n+1}{2}}\, \psi_n(\xi) \right) d\xi \right]$$

$$= -i\hbar \left[ \frac{n}{2} - \frac{n+1}{2} \right] = \frac{i\hbar}{2}$$

$$\langle n | \hat{X}\hat{P}_x \underbrace{- \hat{P}_x \hat{X} + \hat{P}_x \hat{X}}_{} | n \rangle = \langle n | [\hat{X}, \hat{P}_x] + \hat{P}_x \hat{X} | n \rangle$$

$$= i\hbar \langle n | n \rangle + \langle n | \hat{P}_x \hat{X} | n \rangle \quad \Rightarrow$$

$$\frac{i\hbar}{2} = i\hbar + \langle n | \hat{P}_x \hat{X} | n \rangle \Rightarrow \langle n | \hat{P}_x \hat{X} | n \rangle = -\frac{i\hbar}{2}$$

5. Naći disperzije opservabli $\hat{x}$ i $\hat{p}_x$ LHO-a u stanju $\psi_n$. Proveriti odgovarajuću relaciju neodređenosti.

Smeru A ovo je verba!

$\Delta \hat{x}$ za domaći: već je rešeno u nekom od prethodnih zadataka.

$$\langle \hat{p}_x \rangle = \int_{-\infty}^{+\infty} \psi_n^*(\xi) \left( -i\hbar \frac{\partial}{\partial x} \right) \psi_n(\xi)\, d\xi$$

$$= -i\hbar \int_{-\infty}^{+\infty} \psi_n^*(\xi) \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial x} \psi_n(\xi)\, d\xi$$

$$= -i\hbar\sqrt{\alpha} \int_{-\infty}^{+\infty} \psi_n^*(\xi) \frac{\partial \psi_n(\xi)}{\partial \xi}\, d\xi$$

$$\boxed{\frac{\partial}{\partial \xi} \psi_n(\xi) = \sqrt{\frac{n}{2}}\, \psi_{n-1}(\xi) - \sqrt{\frac{(n+1)}{2}}\, \psi_{n+1}(\xi)}$$

$$= -i\hbar\sqrt{\alpha} \int_{-\infty}^{+\infty} \psi_n(\xi) \left( \sqrt{\frac{n}{2}}\, \psi_{n-1}(\xi) - \sqrt{\frac{n+1}{2}}\, \psi_{n+1}(\xi) \right) d\xi$$

$$= -i\hbar\sqrt{\alpha} \sqrt{\frac{n}{2}} \int_{-\infty}^{+\infty} \psi_n(\xi)\psi_{n-1}(\xi)\, d\xi +$$

$$i\hbar\sqrt{\alpha} \sqrt{\frac{n+1}{2}} \int_{-\infty}^{+\infty} \psi_n(\xi)\, \psi_{n+1}(\xi)\, d\xi = 0$$

$$\langle \hat{p}_x^2 \rangle = ?$$

$$\hat{p}_x \to -i\hbar \frac{\partial}{\partial x} \to -i\hbar \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial x} = -i\hbar \sqrt{\alpha} \frac{\partial}{\partial \xi}$$

$$\hat{p}_x^2 \to -\hbar^2 \alpha \frac{\partial^2}{\partial \xi^2}$$

$$\langle \hat{p}_x^2 \rangle = -\hbar^2 \alpha \int_{-\infty}^{+\infty} \psi_n^*(\xi) \frac{\partial^2}{\partial \xi^2} \psi_n^*(\xi)\, d\xi$$

$$= -\hbar^2 \alpha \int_{-\infty}^{+\infty} \psi_n^*(\xi) \frac{\partial}{\partial \xi} \left( \sqrt{\frac{n}{2}} \psi_{n-1}(\xi) - \sqrt{\frac{n+1}{2}} \psi_{n+1}(\xi) \right) d\xi$$

$$= -\hbar^2 \alpha \Bigg[ \sqrt{\frac{n}{2}} \int_{-\infty}^{+\infty} \psi_n(\xi) \frac{\partial}{\partial \xi} \psi_{n-1}(\xi)\, d\xi \; \cancel{+}$$

$$\sqrt{\frac{n+1}{2}} \int_{-\infty}^{+\infty} \psi_n(\xi) \frac{\partial}{\partial \xi} \psi_{n+1}(\xi)\, d\xi \Bigg]$$

$$= -\hbar^2 \alpha \Bigg[ \sqrt{\frac{n}{2}} \int_{-\infty}^{+\infty} \psi_n(\xi) \left( \sqrt{\frac{n-1}{2}} \psi_{n-2}(\xi) - \sqrt{\frac{n}{2}} \psi_n(\xi) \right) d\xi -$$

$$- \sqrt{\frac{n+1}{2}} \int_{-\infty}^{+\infty} \psi_n(\xi) \left( \sqrt{\frac{n+1}{2}} \psi_n(\xi) - \sqrt{\frac{n+2}{2}} \psi_{n+2}(\xi) \right) d\xi \Bigg]$$

$$= -\hbar^2 \alpha \left[ -\frac{n}{2} - \frac{n+1}{2} \right] = \frac{\hbar^2 \alpha}{2} (2n+1)$$

$$\left. \begin{aligned} \Delta \hat{x} &= \sqrt{\frac{2n+1}{2\alpha}} \\ \Delta \hat{p}_x &= \hbar \sqrt{\frac{\alpha}{2} (2n+1)} \end{aligned} \right\} \Rightarrow \Delta \hat{x} \cdot \Delta \hat{p}_x = \frac{\hbar}{2} (2n+1) \geq \frac{\hbar}{2}$$

$$\boxed{\begin{aligned} &\text{Za } n = 0 \\ &\Delta \hat{x} \cdot \Delta \hat{p}_x = \frac{\hbar}{2} \end{aligned}}$$

6. Naći očekivane vrednosti od $\hat{\vec{r}}^2$ u prvom pobuđenom stanju DIHO-a, zadatim kv. brojem $n_x = 1$

$$E_n = (n+1)\hbar\omega \quad , \quad n = n_x + n_y \Rightarrow 1 = \begin{cases} 0+1 \\ 1+0 \end{cases}$$

$$|\psi_{n_x=1}\rangle\,|\psi_{n_y=0}\rangle$$

$$|\psi_{n_x=0}\rangle\,|\psi_{n_y=1}\rangle$$

$$\langle \hat{\vec{r}}^2 \rangle = \langle \psi_{n_x=1}|\langle\psi_{n_y=0}|\,\hat{\vec{r}}^2\,|\psi_{n_x=1}\rangle|\psi_{n_y=0}\rangle$$

$$= \langle \psi_{n_x=1}|\langle\psi_{n_y=0}|\,\hat{x}^2+\hat{y}^2\,|\psi_{n_x=1}\rangle|\psi_{n_y=0}\rangle$$

$$= \langle \psi_{n_x=1}|\langle\psi_{n_y=0}|\,\hat{x}^2\otimes\hat{I}_y + \hat{I}_x\otimes\hat{y}^2\,|\psi_{n_x=1}\rangle|\psi_{n_y=0}\rangle$$

$$= \langle \psi_{n_x=1}|\hat{x}^2|\psi_{n_x=1}\rangle + \langle\psi_{n_y=0}|\hat{y}^2|\psi_{n_y=0}\rangle$$

V. zadatak 2.

$$\langle \hat{x}^2 \rangle = \frac{2n+1}{2\alpha}$$

$$= \frac{2+1}{2\alpha} + \frac{0+1}{2\alpha} = \frac{2}{\alpha}$$

Za pismeni: Očekivane vrednosti od

$\hat{x}\hat{y}$, $\hat{p}_x\hat{p}_y$, $\hat{x}\hat{p}_y$, $\hat{y}\hat{p}_x$, $\hat{x}\hat{p}_x$, $\hat{y}\hat{p}_y$, $\hat{\vec{p}}^2$

a varijacija na temu za $n_y = 1$

7. Naći očekivane vrednosti opservabli $\vec{r}^2$ i $\hat{\vec{p}}$ u prvom pobuđenom stanju TIHO-a, zadatom kvantnim brojevima a) $n_x = 1$, b) $n_z = 1$

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a) Energija TIHO-a $\quad E_n = \left(n + \frac{3}{2}\right)\hbar\omega$

$n = n_x + n_y + n_z$

$n = 1$

$n_x = 1 \quad n_y = 0 \quad n_z = 0 \qquad |\psi_{n_x=1}\rangle|\psi_{n_y=0}\rangle|\psi_{n_z=0}\rangle$

$n_x = 0 \quad n_y = 1 \quad n_z = 0$

$n_x = 0 \quad n_y = 0 \quad n_z = 1$

$$\langle \hat{\vec{r}}^2 \rangle = \langle \psi | \hat{\vec{r}}^2 | \psi \rangle = \langle \psi | \hat{x}^2 + \hat{y}^2 + \hat{z}^2 | \psi \rangle =$$

$$\langle \psi_{n_x=1} | \hat{x}^2 \otimes \hat{I}_y \otimes \hat{I}_z | \psi_{n_x=1} \rangle + \langle \psi_{n_y=0} | \hat{I}_x \otimes \hat{y}^2 \otimes \hat{I}_z | \psi_{n_y=0} \rangle$$

$$+ \langle \psi_{n_z=0} | \hat{I}_x \otimes \hat{I}_y \otimes \hat{z}^2 | \psi_{n_z=0} \rangle =$$

$$\boxed{\begin{array}{l} \text{V. zadatak 2} \\ \langle \hat{x}^2 \rangle = \dfrac{2n+1}{2\alpha} \end{array}}$$

$$= \frac{2\cdot 1 + 1}{2\alpha} + \frac{2\cdot 0 + 1}{2\alpha} + \frac{2\cdot 0 + 1}{2\alpha} = \frac{5}{2\alpha}$$

Ostalo za domaći

Za ispit: Varijacija kao u zadatku 6!

8. Izračunati najverovatniju vrednost koordinate $\hat{X}$ u stanju

a) $\psi_{n=1}(\xi)$

b) $\psi_{n=2}(\xi)$ (domaći ili pismeni)

LHO-a

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$$\psi_n(\xi) = c_n \, e^{-\xi^2/2} H_n(\xi)$$

$$\psi_1(\xi) = c_1 e^{-\xi^2/2} H_1(\xi) = 2c_1 \xi e^{-\xi^2/2}$$

$$W(\hat{\xi}, |\psi_1\rangle, \xi \in [\alpha, \beta]) = \langle \psi_1 | \hat{P}_{[\alpha,\beta]} | \psi_1 \rangle$$

$$= \int_\alpha^\beta |\psi_1(\xi)|^2 d\xi = \int_\alpha^\beta g_r(\xi) \, d\xi$$

$$g_r(\xi) = 4c_1^2 \xi^2 e^{-\xi^2}$$

$$\frac{dg_r(\xi)}{d\xi} = 4c_1^2 \left( 2\xi e^{-\xi^2} + \xi^2(-2\xi) e^{-\xi^2} \right)$$

$$= 4c_1^2 e^{-\xi^2} (2\xi - 2\xi^3)$$

$$= 8c_1^2 e^{-\xi^2} (\xi - \xi^3)$$

$$\frac{dg_r(\xi)}{d\xi} = 0$$

$\xi - \xi^3 = 0 \quad \Rightarrow \quad \xi(1-\xi^2) = 0$

$\xi = 0$

$\xi = \pm 1$

$$\psi_1(\xi) = c_1 e^{-\xi^2/2} H_1(\xi)$$

$$= c_1 e^{-\xi^2/2} 2\xi$$

$$\boxed{g_r(\xi) = 4c_1^2 e^{-\xi^2} \xi^2}$$

$g_r(0) = 0$

$g_r(\pm 1) = 4c_1^2 e^{-1}$

Dakle, vrednosti za $\xi = \pm 1$ su najverovatnije vrednosti položaja, odnosno $x = \pm \frac{1}{\sqrt{\alpha}}$

9. Izračunati najverovatniju vrednost opservable $\hat{y}$ u 1. pobuđenom stanju DIHO-a (određenom kvantnim brojem $n_x=1$) ako je DIHO „pomeren" duž y-ose, ulevo za $y_0$.

$E_n = (n+1)\hbar\omega$, $n = n_x + n_y$

I pobuđeno stanje $\quad n=1 \Rightarrow n_x + n_y = 1$

$|\psi_{n_x=1}\rangle |\psi_{n_y=0}\rangle \quad$ stanje od interesa

$\xi = \sqrt{\alpha}\, x$, $\eta = \sqrt{\alpha}(y-y_0)$

$$\psi(\xi,\eta) = \psi_{n_x=1}(\xi)\,\psi_{n_y=0}(\eta)$$

$$g_v(\xi,\eta) = |\psi(\xi,\eta)|^2 = |\psi_{n_x=1}(\xi)|^2 |\psi_{n_y=0}(\eta)|^2$$

$$g_v(\eta) = \int_{-\infty}^{+\infty} g_v(\xi,\eta)\,d\xi = \int_{-\infty}^{+\infty} |\psi_{n_x=1}(\xi)|^2 |\psi_{n_y=0}(\eta)|^2 d\xi$$

$$= |\psi_{n_y=0}(\eta)|^2 \underbrace{\int_{-\infty}^{+\infty} |\psi_{n_x=1}(\xi)|^2 d\xi}_{1}$$

$$\psi_n(\eta) = C_n e^{-\frac{\eta^2}{2}} H_n(\eta)$$

$$\psi_{n_y=0}(\eta) = C_0 e^{-\frac{\eta^2}{2}} H_0(\eta) = C_0 e^{-\frac{\eta^2}{2}} \Rightarrow |\psi_{n_y=0}(\eta)|^2 = C_0^2 e^{-\eta^2}$$

$g_v'(\eta) = -2\eta C_0^2 e^{-\eta^2}$, $g_v'(\eta) = 0 \Rightarrow \eta = 0 \Rightarrow y - y_0 = 0$

$\boxed{y = y_0}$ najverovatnija vrednost

10. Zadat je operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + i\sqrt{\frac{1}{2m\omega\hbar}}\,\hat{p}_x$$

Pokazati da važi

$$[\hat{a}, \hat{a}^+] = \hat{I}.$$

Zatim eksplicitnim računom (delovanjem na odgovarajuće Ermitove f-je) u koordinatnoj reprezentaciji proveriti rezultate dobijene ranije, tj. da važi:

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

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$$\alpha = \frac{m\omega}{\hbar}$$

$$\hat{a} = \sqrt{\frac{\alpha}{2}}\,\hat{x} + \frac{i}{\hbar}\sqrt{\frac{1}{2\alpha}}\,\hat{p}_x$$

$$\hat{a}^+ = \sqrt{\frac{\alpha}{2}}\,\hat{x} - \frac{i}{\hbar}\sqrt{\frac{1}{2\alpha}}\,\hat{p}_x$$

$$[\hat{a}, \hat{a}^+] = \left[\sqrt{\frac{\alpha}{2}}\,\hat{x} + \frac{i}{\hbar}\sqrt{\frac{1}{2\alpha}}\,\hat{p}_x, \sqrt{\frac{\alpha}{2}}\,\hat{x} - \frac{i}{\hbar}\sqrt{\frac{1}{2\alpha}}\,\hat{p}_x\right]$$

$$= -\frac{i}{\hbar}\sqrt{\frac{\alpha}{2}}\sqrt{\frac{1}{2\alpha}}\,[\hat{x}, \hat{p}_x] + \frac{i}{\hbar}\sqrt{\frac{1}{2\alpha}}\sqrt{\frac{\alpha}{2}}\,[\hat{p}_x, \hat{x}] =$$

$$= -\frac{i}{\hbar}\cdot\frac{1}{2}i\hbar + \frac{i}{\hbar}(-i\hbar)\cdot\frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

Koordinatna reprezentacija

$$a = \sqrt{\frac{\alpha}{2}}\, x + \frac{i}{\hbar}\sqrt{\frac{1}{2\alpha}}\left(-i\hbar \frac{d}{dx}\right)$$

$$\left[\ \xi = \sqrt{\alpha}\, x \ , \quad \frac{d}{dx} = \frac{d}{d\xi}\frac{d\xi}{dx} = \sqrt{\alpha}\frac{d}{d\xi}\ \right]$$

$$\boxed{a = \sqrt{\frac{\alpha}{2}}\,\frac{\xi}{\sqrt{\alpha}} + \sqrt{\frac{1}{2\alpha}}\sqrt{\alpha}\frac{d}{d\xi}}$$

$$a = \frac{\xi}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{d}{d\xi} = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right)$$

$$a^+ = \sqrt{\frac{\alpha}{2}}\, x - \frac{i}{\hbar}\sqrt{\frac{1}{2\alpha}}\left(-i\hbar\frac{d}{dx}\right)$$

$$a^+ = \sqrt{\frac{\alpha}{2}}\frac{\xi}{\sqrt{\alpha}} - \frac{1}{\sqrt{2\alpha}}\sqrt{\alpha}\frac{d}{d\xi}$$

$$= \frac{1}{2}\left(\xi - \frac{d}{d\xi}\right)$$

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$$a\,\psi_n(\xi) = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right)\psi_n(\xi) = \frac{1}{\sqrt{2}}\left(\xi\,\psi_n(\xi) + \frac{d\psi_n(\xi)}{d\xi}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cancel{\sqrt{\frac{n+1}{2}}\,\psi_{n+1}(\xi)} + \sqrt{\frac{n}{2}}\,\psi_{n-1}(\xi) - \cancel{\sqrt{\frac{n+1}{2}}\,\psi_{n+1}(\xi)} + \sqrt{\frac{n}{2}}\,\psi_{n-1}(\xi)\right)$$

$$= \frac{1}{\sqrt{2}}\, 2\sqrt{\frac{n}{2}}\,\psi_{n-1}(\xi) = \sqrt{n}\,\psi_{n-1}(\xi)$$

$$\boxed{a_n\,\psi_n(\xi) = \sqrt{n}\,\psi_{n-1}(\xi)}$$

ostatak za vježbu

11. Rešiti stacionarnu Šredingerovu j-nu za naelektrisanu česticu mase m u spoljašnjem magnetnom polju zadatim vektorskim potencijalom $\vec{A} = (0, B\cdot x, 0)$ gde je B konstanta sa dimenzijama magnetne indukcije. (V. 264. str. Yung Kuo Lim)

NE KTF

$$\vec{\underline{P}} = \vec{P} - \frac{e}{c}\vec{A}(\vec{r})$$ "minimalna preskripcija"

$$\hat{H} = \frac{\hat{\vec{\underline{P}}}^2}{2m} = \frac{1}{2m}\left(\hat{\vec{P}} - \frac{e}{c}\hat{\vec{A}}(\hat{\vec{r}})\right)^2$$

$$\hat{\vec{\underline{P}}} = \hat{\vec{P}} - \frac{e}{c}A(\hat{\vec{r}}) = (\hat{P}_x, \hat{P}_y, \hat{P}_z) - \frac{e}{c}(0, B\cdot\hat{x}, 0)$$

$$= \left(\hat{P}_x, \hat{P}_y - \frac{e}{c}B\hat{x}, \hat{P}_z\right)$$

$$\hat{\vec{\underline{P}}}^2 = \hat{P}_x^2 + \left(\hat{P}_y - \frac{e}{c}B\hat{x}\right)^2 + \hat{P}_z^2$$

$$\hat{P}_x = -i\hbar\frac{\partial}{\partial x}, \quad \hat{P}_y = -i\hbar\frac{\partial}{\partial y}, \quad \hat{P}_z = -i\hbar\frac{\partial}{\partial z}$$

$$\hat{P}_x^2 = -\hbar^2\frac{\partial^2}{\partial x^2}, \quad \hat{P}_y^2 = -\hbar^2\frac{\partial^2}{\partial y^2}, \quad \hat{P}_z^2 = -\hbar^2\frac{\partial^2}{\partial z^2}$$

$$[\hat{H}, \hat{P}_y] = 0, \quad [\hat{H}, \hat{P}_z] = 0 \qquad (1)$$

$$\left[\frac{\hat{\vec{\underline{P}}}^2}{2m}, \hat{P}_y\right] = \frac{1}{2m}[\hat{P}_x^2, \hat{P}_y] + \frac{1}{2m}\left[\left(\hat{P}_y - \frac{e}{c}B\hat{x}\right)^2, \hat{P}_y\right] +$$

$$\frac{1}{2m}[\hat{P}_z^2, \hat{P}_y] = 0$$

Zbog (1) rešenje možemo tražiti preko svojstvenih f-ja za $\hat{p}_y$ i $\hat{p}_z$

J-na koju rešavamo

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2}+\left(-i\hbar\frac{\partial}{\partial y}-\frac{e}{c}Bx\right)^2+\hbar^2\frac{\partial^2}{\partial z^2}\right)\psi(\vec{r})=E\psi(\vec{r}) \quad (2)$$

$$\psi(\vec{r})=\psi(x)\phi(y)\chi(z)$$

$$\phi(y)=\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}yp_y}, \quad \chi(z)=\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{i}{\hbar}zp_z} \quad (3)$$

(3) → (2)

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2}+\left(-i\hbar\frac{\partial}{\partial y}-\frac{e}{c}Bx\right)^2-\hbar^2\frac{\partial^2}{\partial z^2}\right)\psi(x)\phi(y)\chi(z)=$$

$$=E\,\psi(x)\phi(y)\chi(z) \quad (4)$$

L.S. od (4)

$$-\frac{\hbar^2}{2m}\phi(y)\chi(z)\frac{\partial^2\psi(x)}{\partial x^2}+\frac{\psi(x)\chi(z)}{2m}\left(-i\hbar\frac{\partial}{\partial y}-\frac{e}{c}Bx\right)^2\phi(y)-$$

$$-\frac{\hbar^2}{2m}\psi(x)\phi(y)\frac{\partial^2\chi(z)}{\partial z^2}$$

$$\frac{\partial\chi(z)}{\partial z}=\frac{1}{\sqrt{2\pi\hbar}}\left(\frac{i}{\hbar}\right)p_z\,e^{\frac{i}{\hbar}zp_z}$$

$$\frac{\partial^2\chi(z)}{\partial z^2}=\frac{1}{\sqrt{2\pi\hbar}}\left(\frac{i}{\hbar}\right)^2p_z^2\,e^{\frac{i}{\hbar}zp_z}=-\frac{p_z^2}{\sqrt{2\pi\hbar}}\frac{e^{\frac{i}{\hbar}zp_z}}{\hbar^2}=-\frac{p_z^2}{\hbar^2}\chi(z)$$

$$\left(-i\hbar \frac{\partial}{\partial y} - \frac{e}{c}Bx\right)^2 \phi(y) = \left(-i\hbar \frac{\partial}{\partial y} - \frac{e}{c}Bx\right)\left(-i\hbar \frac{\partial}{\partial y} - \frac{e}{c}Bx\right)\phi$$

$$= \left(-i\hbar \frac{\partial}{\partial y} - \frac{e}{c}Bx\right)\left(-i\hbar \frac{\partial \phi(y)}{\partial y} - \frac{e}{c}Bx\,\phi(y)\right) =$$

$$= -\hbar^2 \frac{\partial^2 \phi(y)}{\partial y^2} + 2i\hbar \frac{e}{c}Bx \frac{\partial \phi(y)}{\partial y} + \frac{e^2}{c^2}B^2x^2\phi(y) =$$

$$\phi(y) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar} y p_y}$$

$$\frac{\partial^2 \phi(y)}{\partial y^2} = \left(\frac{1}{\sqrt{2\pi\hbar}} \frac{i}{\hbar} p_y\, e^{\frac{i}{\hbar} y p_y}\right)'_y = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{i}{\hbar} p_y\right)^2 e^{\frac{i}{\hbar} y p_y} =$$

$$= -\frac{1}{\sqrt{2\pi\hbar}} \frac{p_y^2}{\hbar^2} e^{\frac{i}{\hbar} y p_y} = -\frac{p_y^2}{\hbar^2}\phi(y)$$

$$\frac{\partial \phi(y)}{\partial y} = \frac{1}{\sqrt{2\pi\hbar}} \frac{i}{\hbar} p_y\, e^{\frac{i}{\hbar} y p_y} = \frac{i}{\hbar} p_y\, \phi(y)$$

$$= -\hbar^2\left(-\frac{p_y^2}{\hbar^2}\right)\phi(y) + 2i\hbar \frac{e}{c}Bx \frac{i}{\hbar} p_y\, \phi(y) + \frac{e^2}{c^2}B^2x^2\phi(y)$$

$$= p_y^2\, \phi(y) - 2\frac{e}{c}Bx\, p_y\, \phi(y) + \frac{e^2}{c^2}B^2x^2\phi(y)$$

$$= \left(p_y - \frac{e}{c}Bx\right)^2 \phi(y)$$

Dalje

$$-\hbar^2 \phi(y)\chi(z)\frac{\partial^2\psi(x)}{\partial x^2} + \psi(x)\chi(z)\left(p_y - \frac{e}{c}Bx\right)^2\phi(y) + p_z^2$$

$$\psi(x)\phi(y)\chi(z)$$

odnosno

$$\left(-\hbar^2 \frac{\partial^2\psi(x)}{\partial x^2} + \left(p_y-\frac{e}{c}Bx\right)^2\psi(x) + p_z^2\psi(x)\right)\phi(y)\chi(z)$$

a to je jednako DS od (4)

pa se dobija

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2}+\frac{1}{2m}\left(p_y-\frac{e}{c}Bx\right)^2\psi(x) = \underbrace{\left(E-\frac{p_z^2}{2m}\right)}_{E'}\psi(x)$$

$$\frac{1}{2m}\left(p_y - \frac{e}{c}Bx\right)^2 = \frac{1}{2m}\left(\frac{eB}{c}\right)^2\left(\frac{p_y c}{eB}-x\right)^2 =$$

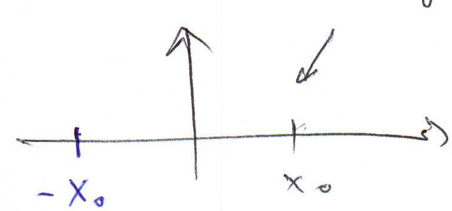
$$= \frac{1}{2} m \left(\frac{eB}{mc}\right)^2 \left(x-\frac{p_y c}{eB}\right)^2$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + \frac{1}{2}m\left(\frac{eB}{mc}\right)^2\left(x-\frac{p_y c}{eB}\right)^2\psi(x) = E'\psi(x) \quad (5)$$

Pomereni LHO

$$\hat H = \frac{1}{2m}\hat p_x^2 + \frac{1}{2}m\omega^2(\hat x - x_0)^2$$

osciluje oko $x_0$



Dif. j-na za pomereni LHO ( stacionarna SJ

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi_n(x)}{\partial x^2}+\frac{1}{2}m\omega^2(x-x_0)^2\Psi_n(x)=E_n\Psi_n(x) \quad (*)$$

sv. f-je $\Psi_n(x-x_0)$, $E_n=(n+\frac{1}{2})\hbar\omega$

Uporedivsi (5) sa (*) jasno je da je (5) dif. j-na za pomereni LHO sa

$$\omega=\frac{eB}{mc} \quad i \quad x_0=\frac{p_y c}{eB}$$

$$\Psi(\vec{r})=\Psi_n\left(x-\frac{p_y c}{eB}\right)\phi(y)\chi(z)$$

$$E_n'=E_n-\frac{p_z^2}{2m}, \quad E_n=(n+\frac{1}{2})\hbar\omega$$

Varijacija na temu uz pomoć raznih vektorskih potencijala ! Ispit

$\vec{A}=(Bx,0,0)$ $\quad\vec{A}(0,0,Bx)$

Da li može ?

$\vec{A}=(Bx,By,0)\rightarrow$ Da li bi se svodilo na DiHO?

A varijacija tipa

$\vec{A}=(0,Bx^2,0)$ ?

12. Naći verovatnoću da se merenjem energije LHO-a u stanju $\psi(\xi) = A e^{-(\xi-\xi_0)^2/2}$ dobije vrednost koja odgovara kvantnom broju n. Iskoristiti relaciju $\int_{-\infty}^{+\infty} e^{-\xi^2} H_m(\xi) H_n(\xi)\, d\xi = 2^n n! \sqrt{\pi}\, \delta_{mn}$.

$$W(\hat{H}, \psi, E_n) = \langle \psi | \hat{P}_n | \psi \rangle = \langle \psi | \psi_n \rangle \langle \psi_n | \psi \rangle$$

$$= |\langle \psi_n | \psi \rangle|^2$$

$$\langle \psi_n | \psi \rangle = \int_{-\infty}^{+\infty} \psi_n(\xi)\, \psi(\xi)\, d\xi$$

$$\psi_n(\xi) = c_n e^{-\xi^2/2} H_n(\xi)$$

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n e^{-\xi^2}}{d\xi^n}$$

$$\psi(\xi) = A e^{-\frac{(\xi-\xi_0)^2}{2}}$$

$$\xi_0 = 2\lambda \quad \leftarrow \text{TRIK}$$

$$\psi(\xi) = e^{-\frac{(\xi-2\lambda)^2}{2}}$$

$$= e^{-\frac{\xi^2}{2}} e^{2\lambda\xi} e^{-2\lambda^2}$$

$$\psi(\xi) = e^{-\frac{\xi^2}{2}} e^{-\lambda^2} e^{-\lambda^2} e^{2\lambda\xi}$$

$$= e^{-\left(\frac{\xi^2}{2}+\lambda^2\right)} e^{-\lambda^2+2\lambda\xi}$$

$$e^{-\lambda^2+2\lambda\xi} = \sum_{n=0}^{\infty} H_n(\xi)\,\frac{\lambda^n}{n!}$$

$$\langle\Psi|\Psi_n\rangle = \int_{-\infty}^{+\infty} \psi(\xi)\,\psi_n(\xi)\,d\xi$$

$$= A\int_{-\infty}^{+\infty} e^{-\left(\frac{\xi^2}{2}+\lambda^2\right)}\, e^{-\lambda^2+2\lambda\xi}\, C_n e^{-\xi^2/2} H_n(\xi)\,d\xi$$

$$= AC_n \int_{-\infty}^{+\infty} e^{-\frac{\xi^2}{2}+\lambda^2-\frac{\xi^2}{2}} \sum_{m=0}^{\infty} H_m(\xi)\frac{\lambda^m}{m!} H_n(\xi)\,d\xi$$

$$= AC_n e^{-\lambda^2} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \underbrace{\int_{-\infty}^{+\infty} e^{-\xi^2} H_m(\xi) H_n(\xi)\,d\xi}_{\text{Mat. fiz.} \; = 2^n n! \sqrt{\pi}\,\delta_{mn}}$$

$$= AC_n e^{-\lambda^2} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!}\, 2^n n!\, \delta_{mn} \cdot \sqrt{\pi}$$

$$= AC_n e^{-\lambda^2} \sqrt{\pi}\, \frac{\lambda^n}{\cancel{n!}}\, 2^n \cancel{n!}$$

$$= AC_n \sqrt{\pi}\, 2^n \lambda^n e^{-\lambda^2}$$

$$W(\ldots) = |\langle\Psi|\Psi_n\rangle|^2 = A^2 C_n^2 \pi\, 2^{2n} \lambda^{2n} e^{-2\lambda^2}$$

Koliko je A?

13. Naći disperziju opservable $\hat{x}^2$ ($\hat{\sigma}_x^2$) u stanjima

a) $|n=2\rangle$, LHO-a

Ovo je ispitni zadatak, nije za vezbe!

b) $|n_x=0\rangle |n_y=1\rangle$, DIHO-a ME KTF

c) $|n_x=0\rangle |n_y=0\rangle |n_z=1\rangle$, TIHO-a (za pismeni)

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a) $|n=2\rangle$

$$\Delta\hat{x}^2 = \sqrt{\langle\hat{x}^4\rangle - \langle\hat{x}^2\rangle^2}$$

V. zadatak 2

$$\langle\hat{x}^2\rangle = \frac{2n+1}{2\alpha}$$

$$\langle\hat{x}^4\rangle = \langle\psi_n|\hat{x}^4|\psi_n\rangle = \int_{-\infty}^{+\infty}\psi_n^*(\xi)\, x^4\, \psi_n(\xi)\, d\xi$$

$$\left[\quad \xi = \sqrt{\alpha}\, x, \quad x = \frac{\xi}{\sqrt{\alpha}} \quad\right]$$

$$= \frac{1}{\alpha^2}\int_{-\infty}^{+\infty}\psi_n(\xi)\, \xi^4\, \psi_n(\xi)\, d\xi$$

$$\xi^4\psi_n(\xi) = ?$$

$$\boxed{\xi\psi_n(\xi) = \sqrt{\frac{n+1}{2}}\,\psi_{n+1}(\xi) + \sqrt{\frac{n}{2}}\,\psi_{n-1}(\xi)}$$

$$\xi^2\psi_n(\xi) = \sqrt{\frac{n+1}{2}}\,\xi\psi_{n+1}(\xi) + \sqrt{\frac{n}{2}}\,\xi\psi_{n-1}(\xi)$$

$$\xi^2 \psi_n(\xi) = \sqrt{\frac{n+1}{2}} \left( \sqrt{\frac{n+2}{2}} \psi_{n+2}(\xi) + \sqrt{\frac{n+1}{2}} \psi_n(\xi) \right) +$$

$$+ \sqrt{\frac{n}{2}} \left( \sqrt{\frac{n}{2}} \psi_n(\xi) + \sqrt{\frac{n-1}{2}} \psi_{n-2}(\xi) \right)$$

$$\xi^2 \psi_n(\xi) = \sqrt{\frac{(n+1)(n+2)}{4}} \psi_{n+2}(\xi) + \frac{n+1}{2} \psi_n(\xi) +$$

$$\frac{n}{2} \psi_n(\xi) + \sqrt{\frac{n(n-1)}{4}} \psi_{n-2}(\xi)$$

$$\xi^2 \psi_n(\xi) = \sqrt{\frac{(n+1)(n+2)}{4}} \psi_{n+2}(\xi) + \frac{2n+1}{2} \psi_n(\xi) + \sqrt{\frac{n(n-1)}{4}} \psi_{n-2}(\xi$$

$$\xi^3 \psi_n(\xi) = \sqrt{\frac{(n+1)(n+2)}{4}} \xi \psi_{n+2}(\xi) + \frac{2n+1}{2} \xi \psi_n(\xi) + \sqrt{\frac{n(n-1)}{4}} \xi \psi_{n-2}(\xi)$$

$$= \sqrt{\frac{(n+1)(n+2)}{4}} \left( \sqrt{\frac{n+3}{2}} \psi_{n+3}(\xi) + \sqrt{\frac{n+2}{2}} \psi_{n+1}(\xi) \right) +$$

$$\frac{2n+1}{2} \left( \sqrt{\frac{n+1}{2}} \psi_{n+1}(\xi) + \sqrt{\frac{n}{2}} \psi_{n-1}(\xi) \right) +$$

$$+ \sqrt{\frac{n(n-1)}{4}} \left( \sqrt{\frac{n-1}{2}} \psi_{n-1}(\xi) + \sqrt{\frac{n-2}{2}} \psi_{n-3}(\xi) \right)$$

$$= \sqrt{\frac{(n+1)(n+2)(n+3)}{8}} \psi_{n+3}(\xi) + (n+2) \sqrt{\frac{n+1}{8}} \psi_{n+1}(\xi)$$

$$\frac{2n+1}{2} \sqrt{\frac{n+1}{2}} \psi_{n+1}(\xi) + \frac{2n+1}{2} \sqrt{\frac{n}{2}} \psi_{n-1}(\xi) + (n-1) \sqrt{\frac{n}{8}} \psi_{n-1}(\xi$$

$$+ \sqrt{\frac{n(n-1)(n-2)}{8}} \psi_{n-3}(\xi)$$

$$= \sqrt{\frac{(n+1)(n+2)(n+3)}{8}}\,\psi_{n+3}(\xi) + \frac{(n+2)}{2}\sqrt{\frac{n+1}{2}}\,\psi_{n+1}(\xi) + \frac{2n+1}{2}\sqrt{\frac{n+1}{2}}\,\psi_{n+1}(\xi$$

$$+ \frac{2n+1}{2}\sqrt{\frac{n}{2}}\,\psi_{n-1}(\xi) + \frac{n-1}{2}\sqrt{\frac{n}{2}}\,\psi_{n-1}(\xi) + \sqrt{\frac{n(n-1)(n-2)}{8}}\,\psi_{n-3}(\xi$$

$$= \sqrt{\frac{(n+1)(n+2)(n+3)}{8}}\,\psi_{n+3}(\xi) + \frac{3n+3}{2}\sqrt{\frac{n+1}{2}}\,\psi_{n+1}(\xi) + \frac{3n}{2}\sqrt{\frac{n}{2}}\,\psi_{n-1}(\xi$$

$$+ \sqrt{\frac{n(n-1)(n-2)}{8}}\,\psi_{n-3}(\xi)$$

$$\xi^4\psi_n(\xi) = \sqrt{\frac{(n+1)(n+2)(n+3)}{8}}\,\xi\psi_{n+3}(\xi) +$$

$$+ \frac{3n+3}{2}\sqrt{\frac{n+1}{2}}\,\xi\psi_{n+1}(\xi) + \frac{3n}{2}\sqrt{\frac{n}{2}}\,\xi\psi_{n-1}(\xi) +$$

$$\sqrt{\frac{n(n-1)(n-2)}{8}}\,\xi\psi_{n-3}(\xi) =$$

$$\sqrt{\frac{(n+1)(n+2)(n+3)}{8}}\left(\sqrt{\frac{(n+4)}{2}}\,\psi_{n+4}(\xi) + \sqrt{\frac{n+3}{2}}\,\psi_{n+2}(\xi)\right) +$$

$$+ \frac{3n+3}{2}\sqrt{\frac{n+1}{2}}\left(\sqrt{\frac{n+2}{2}}\,\psi_{n+2}(\xi) + \sqrt{\frac{n+1}{2}}\,\psi_n(\xi)\right) +$$

$$\frac{3n}{2}\sqrt{\frac{n}{2}}\left(\sqrt{\frac{n}{2}}\,\psi_n(\xi) + \sqrt{\frac{n-1}{2}}\,\psi_{n-2}(\xi)\right) +$$

$$\sqrt{\frac{n(n-1)(n-2)}{8}}\left(\sqrt{\frac{n-2}{2}}\,\psi_{n-2}(\xi) + \sqrt{\frac{n-3}{2}}\,\psi_{n-4}(\xi)\right) =$$

$$= \sqrt{\frac{(n+1)(n+2)(n+3)(n+4)}{16}}\, \psi_{n+4}(\xi) + \frac{n+3}{2}\sqrt{\frac{(n+1)(n+2)}{4}}\, \psi_{n+2}(\xi)$$

$$\frac{3n+3}{2}\sqrt{\frac{(n+1)(n+2)}{4}}\, \psi_{n+2}(\xi) + \frac{3n+3}{2}\,\frac{n+1}{2}\, \psi_n(\xi) + \frac{3n}{2}\,\frac{n}{2}\, \psi_n(\xi)$$

$$+ \frac{3n}{2}\sqrt{\frac{n(n-1)}{4}}\, \psi_{n-2}(\xi) + \frac{(n-2)}{2}\sqrt{\frac{n(n-1)}{4}}\, \psi_{n-2}(\xi) +$$

$$\sqrt{\frac{n(n-1)(n-2)(n-3)}{16}}\, \psi_{n-4}(\xi)$$

Koeficijenti uz $\psi_n(\xi)$

$$\frac{3n+3}{2}\,\frac{n+1}{2} + \frac{3n^2}{4} = \ldots = \frac{3(2n^2+2n+1)}{4}$$

$$\langle \hat{x}^4 \rangle = \frac{1}{\alpha^2}\,\frac{3(2n^2+2n+1)}{4}$$

$$\Delta \hat{x}^2 = \sqrt{\frac{1}{4\alpha^2}\, 3(2n^2+2n+1) - \frac{(2n+1)^2}{4\alpha^2}}$$

$$= \frac{1}{2\alpha}\sqrt{6n^2+6n+3-4n^2-4n-1}$$

$$= \frac{1}{\sqrt{2}\,\alpha}\sqrt{n^2+n+1}$$

Za $n=2$      b) Za domaći

$$\Delta \hat{x}^2 = \sqrt{\frac{7}{2}}\,\alpha^{-1}$$